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SUBJECT MATTER KNOWLEDGE FOR TEACHING AND THE CASE OF FUNCTIONS

ABSTRACT. Interest in teachers' subject matter knowledge has arisen in recent years. But most of the analysis has been general and not topic-specific. This paper shows how one may approach the question of teachers' knowledge about mathematical topics. It demonstrates the building of an analytic framework of subject matter knowledge for teaching a specific topic in mathematics and then uses the concept of function to provide an illustrative case of a paradigm for analyzing subject matter knowledge for teaching. The choice of the aspects, which form the main facets of the framework, was based on integrated knowledge from several bodies of work: the role and importance of the topic in the discipline of mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of mathematical concepts in general and the specific topic in particular; and research and theoretical work on teachers' subject matter knowledge and its role in teaching. An application of the framework in the case of the concept of function is described and illustrated by anecdotes drawn from a study of prospective secondary teachers' knowledge and understanding of functions.

INTRODUCTION

Mathematics educators today are concerned with the way mathematics is taught. They call for making a change in the way teachers teach to emphasize teaching for understanding and meaningful learning (e.g., Davis, 1986; Educational Technology Center, 1988; Lampert, 1988; Lappan and Schram, 1989; NCTM, 1989a; Peterson, 1988; Resnick, 1987; Romberg, 1983; Schoenfeld, 1987). The teacher's role is to help the learner achieve understanding of the subject matter. But in order to do so the teachers themselves need to have solid knowledge of the subject matter. A teacher who has solid mathematical knowledge for teaching is more capable of helping his/her students achieve a meaningful understanding of the subject matter. Subject matter knowledge is only one component of the knowledge of a well prepared teacher – nevertheless – an important one.

Recent reform efforts (e.g., Carnegie Task force, 1986; Holmes Group, 1986; NCTM, 1989b) are designed to improve professional teacher education. One of the goals of the current attempts to reform teaching is to strengthen the subject matter preparation of teachers: "... [prospective teachers] will be expected to pass an examination demonstrating their mastery of the subject they will teach" (Holmes Group, 1986). At the same time, interest in defining and analyzing what subject matter knowledge for teaching means has arisen.

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Conceptions of teachers' subject matter knowledge have changed throughout the years. At the beginning of this century, Dewey (1904) described teacher subject matter knowledge in qualitative terms which did not provide a straightforward way of measuring or evaluating knowledge. When process-product research on teaching became popular, teacher subject matter knowledge was defined in quantitative terms – by the number of courses taken in college or teachers' scores on standardized tests (Ball, in press; Wilson, Shulman, and Richert, 1987). But these "measures" are problematic, since they do not represent teachers' knowledge of the subject matter. Shulman's (1986) Presidential Address at the 1985 annual meeting of the American Educational Research Association in Chicago, signaled a return to a definition of teachers' subject matter knowledge in qualitative terms. Other scholars today also write about teachers' subject matter knowledge in qualitative terms (Ball, in press; Leinhardt and Smith, 1985; Tamir, 1987; Wilson et al., 1987). Defining teachers' subject matter knowledge not by the number of courses they have taken or their success on standardized tests, but by analyzing what it means to know mathematics, has some promise to contribute to the improvement of the quality of subject matter preparation of teachers and therefore the quality of teaching and learning.

Still, analyzing what teachers' subject matter knowledge means in general in mathematics, does not inform us of what subject matter knowledge teachers need to have in order to teach a specific piece of mathematics. While qualitative analysis of teachers' subject matter knowledge has brought us one step forward from a simplistic list of competencies that served as a criterion to knowledge, such analyses miss specific characteristics of knowledge needed for teaching a specific mathematical topic.

This paper shows how one may approach the question of teachers' knowledge about mathematical topics. It demonstrates the building of an analytic framework of subject matter knowledge for teaching a specific topic in mathematics. First, the development of a general framework is discussed, emphasizing the general and guiding principles of the analysis. Then, an application of the framework in the case of the concept of function is described to provide an illustrative case of a paradigm for analyzing subject matter knowledge for teaching. Anecdotes drawn from a cross-institutional study of prospective secondary teachers' knowledge of functions (Even, 1989) are used as illustration for clarifying the framework and pointing out weaknesses in existing teacher knowledge.

THE GENERAL FRAMEWORK

Teachers' subject matter knowledge about a specific mathematical topic is influenced by what they know across different domains of knowledge. Therefore, analysis of teachers' subject matter knowledge about a specific piece of mathematics should integrate several bodies of knowledge; the role and importance of the topic in mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of mathematical concepts in general and the specific topic in particular; and research and theoretical work on teachers' subject matter knowledge and its role in teaching. The analysis should also take into account the specific population of consideration. As a result of this analysis, the following seven aspects seemed to form the main facets of teachers' subject matter knowledge about a specific mathematical topic.

Essential Features

One aspect of the framework deals with the concept image, paying attention to the essence of the concept. Vinner (1983) defines concept image as the mental picture of this concept (i.e., the set of all 'pictures' that have ever been associated with the concept in the person's mind) together with the set of properties associated with the concept (in the person's mind). The image of a concept might be different for different people. Resnick and Ford (1984), following Greeno's (1978) suggestion, present 'correspondence' – the match of one's subjective mental picture of a specific concept with the correct mathematical concept – as an important criterion for evaluating well-structured knowledge about mathematics. Not many will dispute the general claim that teachers should have a good match between their understanding of a specific mathematical concept they teach and the "correct" mathematical concept. But what is the "correct" mathematical concept?

Although the answer to this question is not definite and is content and context specific, we argue that teachers need to be able to judge whether an instance belongs to a concept family by using an analytical judgement as opposed to a mere use of a prototypical judgement. The first type of reasoning is based on the concept's critical attributes (the attributes that an instance must have in order to be a concept example). They are derived from the concept's mathematical definition. The second type uses a prototypical example as the frame of reference either by applying a visual judgement to other instances or by basing the judgement on the prototype's self attributes and imposing them on other concept examples (Hershkowitz, 1990).

The concept image is determined mainly by the instances people meet and not by the concept definition. However, it is not enough that teachers are able to distinguish between concept examples and non-examples when the instances match their concept image only. Teachers, especially when they let their students explore and raise questions, may be put in situations where they have to deal with unfamiliar instances. Their pedagogical decisions – questions they ask, activities they suggest – are based on their subject matter knowledge. Therefore, it is necessary that teachers are able to correctly distinguish between concept examples and non-examples.

In addition to the pedagogical arguments described above there are also cultural arguments. Many mathematical concepts have been changed during the years. They have evolved not because someone arbitrarily decided to make changes but rather because new knowledge in mathematics created the need for the extended concept definition. New discoveries created new branches of mathematics which also led to changes in the definitions. Mathematics teachers who are constrained by their limited and underdeveloped concept image may also be deprived from understanding current mathematics which is based on a more modern conception of functions.

Different Representations

Many important concepts in mathematics appear and behave in different ways. While the pure definition of a concept may make it look narrow and one-sided, complex concepts have different labels and notations and they appear in different representations throughout the numerous divisions of mathematics. Therefore, another aspect of the framework involves the different representations of the concept.

Understanding a concept in one representation does not necessarily mean that one understands it in another representation. Teachers need to understand concepts in different representations, and be able to translate and form linkages among and between them. Different representations give different insights which allow a better, deeper, more powerful and more complete understanding of a concept. When dealing with a mathematical concept in different representations, one may abstract the concept by grasping the common properties of the concept while ignoring the irrelevant characteristics that are imposed by the specific representation at hand (Dufour-Janvier, Bednarz, and Belanger, 1987; Lesh, Post, and Behr, 1987).

Alternative Ways of Approaching

The appearances of a complex concept in various forms, representations, labels and notations are enhanced by the different uses of the concept in the different divisions of mathematics, other disciplines or everyday life. Alternative ways of approaching the same concept are used. These alternative ways are different from one another and none of them are suitable for all situations. Sometimes, when more than one approach can be used, some are more appropriate than others. Therefore, there is a need to make good choices between different available approaches. Teachers should be familiar with the main alternative approaches and their uses.

The Strength of the Concept

The success of a concept in the discipline of mathematics is rooted in the new opportunities it opens. Concepts become important and powerful because there is something special about them which is very unique and opens new possibilities. Teachers should, therefore, have a good understanding of the unique powerful characteristics of the concept. The related important sub-topics or sub-concepts, as with any other concept, cannot be fully understood or appreciated if viewed in one simplistic way only. Understanding such sub-topics or sub-concepts requires knowing the general meaning which captures the essence of the definition as well as a more sophisticated formal mathematical knowledge.

Basic Repertoire

With every mathematical topic or concept there is also a need to know and have easy access to specific examples. The basic repertoire includes powerful examples that illustrate important principles, properties, theorems, etc. Acquiring the basic repertoire gives insights into and a deeper understanding of general and more complicated knowledge. When having to deal with complex situations the basic knowledge serves as a reference, monitoring ways of thinking and acting. The basic repertoire should be well known and familiar in order to be readily available for use. But this is not to suggest that it should be memorized and used without understanding. On the contrary – only if the basic repertoire is acquired meaningfully and with understanding can it be used appropriately and wisely.

Knowledge and Understanding of a Concept

Conceptual knowledge is described by Hiebert and Lefevre (1986) as knowledge which is rich in relationships. It is a network of concepts and relationships (Bell, Costello, and Kuchemann, 1983). The learning of a new concept or relationship implies the addition of a node or link to the existing cognition structure, thus making the whole more stable than before. Meaning, say Hiebert and Lefevre, is generated as relationships between units of knowledge are recognized or created. So, conceptual knowledge must be learned meaningfully. Procedural knowledge, on the other hand, is made up of the formal language of mathematics and the algorithms for completing mathematical tasks. Procedural knowledge can be learned with or without meaning.

School mathematics tends to over-emphasize procedural knowledge without close relation to conceptual knowledge and meaning (e.g., Davis, 1986; Educational Technology Center, 1988; Lampert, 1988; Lappan and Schram, 1989; NCTM, 1989a; Peterson, 1988; Schoenfeld, 1987). Students are asked to memorize a lot of facts and procedures without paying attention to an understanding of concepts and their application even though the advanced technology we have now and will have in the future, will handle this aspect of performing procedures and algorithms.

But our desire to achieve meaningful learning and understanding does not mean that we should ignore procedural knowledge. While there is no reason to memorize algorithms that are easily done by machines, procedural knowledge is still important. Nesher (1986) claims that the dichotomy between learning algorithms and understanding is a superficial and misleading dichotomy since research on mathematical performance does not inform us about the relationship between success in algorithmic performance versus success in understanding nor does it give evidence about the contribution of understanding to algorithmic performance. Resnick and Ford (1984) add that memorization of certain facts and procedures is important not so much as an end in itself, but as a way to extend the capacity of the working memory. This can be done by developing automaticity of responding. When certain processes can be carried out automatically, without need for direct attention, more space becomes available in working memory for processes that do require attention.

Mathematical knowledge in general, should include both kinds of knowledge and the relationships between them. When knowledge is used dynamically to solve a problem or perform some nontrivial task, it is the relationships between conceptual and procedural knowledge that become important (Silver, 1986). People are not competent in mathematics if either kind of knowledge is deficient or if they have been acquired but remain separate entities (Hiebert and Lefevre, 1986). When concepts and procedures are not connected, people may have a good intuitive feel for mathematics but not be able to solve problems, or they may generate answers but not understand what they are doing.

Knowledge about Mathematics

Knowledge of a specific piece of mathematics includes more than conceptual and procedural knowledge. It also includes knowledge about the nature of mathematics. This is a more general knowledge about a discipline which guides the construction and use of conceptual and procedural knowledge. It includes ways, means and processes by which truths are established as well as the relative centrality of different ideas (Ball, in press; Lampert, 1988; Schoenfeld, 1988; Shulman, 1986; Tamir, 1987; Thompson, 1984). The nature of mathematics also includes its everchanging character as well as its being a free invention of the human intellect which is influenced by different forces inside and outside mathematics (Wilder, 1972).

Up to this point, this analysis has focused on teachers' knowledge about mathematical topics. The next part focuses on the implication of the general framework in the case of the concept of function with illustrations of difficulties caused by lack of understandings related to the different aspects. The illustrations are taken from a study of prospective secondary mathematics teachers' knowledge and understanding about functions (Even, 1989). Participants were 162 prospective secondary mathematics teachers in the last stage of their formal preservice preparation at eight midwestern universities in the USA. Data were gathered in two phases from November 1987 to April 1988. During the first phase, the prospective teachers completed an open-ended questionnaire. This questionnaire included non-standard mathematics problems addressing the seven interrelated aspects of function knowledge. The questionnaire also asked respondents to appraise and comment on examples of students' work (each of which represented some misunderstanding or error related to functions). In a second phase of data collection, intensive interviews were conducted with ten of the participating prospective teachers in order to augment the analysis.

THE CASE OF FUNCTIONS

Essential Features – What is a Function?

Freudenthal (1983), in his exhaustive analysis, considers arbitrariness and univalence to be the essential features of the concept of function as it has evolved in history. Arbitrariness is closely related to an analytical judgement whether an instance belongs to a concept family (as described in the general framework) while univalence is a function specific characteristic.

Arbitrariness of functions. The arbitrary nature of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The first means that functions do not have to be described by any specific expression, follow some regularity, or be described by a graph with any particular shape. The function that describes the relationship between time and temperature is an example of this kind of functions. The arbitrary nature of the two sets means that functions do not have to be defined on any specific set of objects; in particular, the sets do not have to be sets of numbers. A rotation of the plane is an example of this type of functions since it is defined on points.

The 18th century mathematicians struggled with the arbitrariness idea of functions. It was not until the 19th century, when Dirichlet introduced the function (well known now as Dirichlet Function) which to each rational number assigns the number 1 and to each irrational – the number 0, that arbitrary functions started to be considered functions and thus the concept enlarged its meaning. Later, in addition to the arbitrariness of the relationship between the variables, the variables themselves or the sets on which the function is defined were allowed to be any sets – arbitrary sets.

The following example shows a rejection of the arbitrariness nature of functions as a result of a use of a prototypical judgement whether an instance is a function, combined with a limited concept image. A prospective teacher, when having to decide whether the following was a function:

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$$g(x) = \begin{cases} x, & \text{if x is a rational number,} \\ 0, & \text{if x is an irrational number.} \end{cases}$$

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based his decision that it was a function on a correct definition of a function: "There is an assignment of a single value to each number." His answer may give the impression that this subject makes his decisions whether something is a function by using the definition for an analytical judgement, and therefore accepts the arbitrary nature of functions. But later, it became clear that things were not so simple. When the prospective teacher was asked to sketch the corresponding graph, he got a few points on the x-axis for irrational numbers: π, $\frac{7}{4}$(!), $\sqrt{13}$; and a diagonal line – $y = x$ – with "holes" in it (see Figure 1). Having an unfamiliar graph in

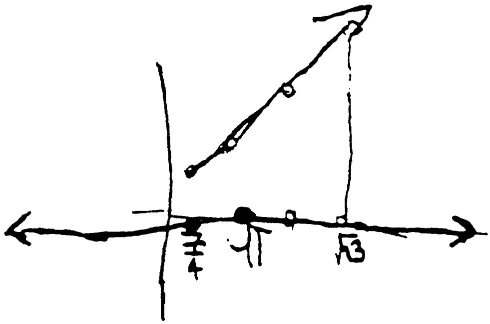

Fig. 1. A function?

front of him, the subject had a hard time deciding whether this graph was a graph of a function. His difficulty was caused by the discrepancy between his concept image of a function and the formal concept definition.

I don't know if it's a function. It fits the criteria of mapping, but it does not look pretty. It's not really graphable. It might just not be. But this is a discontinuous function. You're allowed to do discontinuous. There aren't sharp points. Oh, well . . . [shrugged his shoulders, cannot make up his mind whether the given was a function].

The above situation was not uncommon among the prospective teachers who participated in the study, although it appeared in different forms. Many rejected the arbitrary nature of functions either because they expected functions to always be representable by formulas, graphs of functions to be "nice" and "reasonable", or functions to somehow be "known".

These results agree with the results of other researchers who have investigated high school and college students' ideas about what makes something a function and compared students' concept images (Vinner, 1983) to the mathematical definition (Dreyfus and Eisenberg, 1983, 1987; Lovell, 1971; Markovits, Eylon, and Bruckheimer, 1983, 1986; Marnyanskii, 1975; Thomas, 1975; Vinner, 1983; Vinner and Dreyfus, 1989). The results of these studies point to a limited view of functions which is caused by having some specific expectations about functions and their behavior. This situation is understandable – almost all the functions that high school and sometimes even college students meet are the kind that have a "nice" graph and can be described by a formula, so the students' concept image of a function is determined by the functions they meet and not by the modern definition of a function which emphasizes the arbitrary nature of functions. But while this could be acceptable with students, this is not acceptable with teachers. We cannot accept a situation where secondary teachers at the end of the 20th century have a limited concept of function, similar to the one from the 18th century. Teachers' incomplete conception of functions is problematic and may contribute to the cycle of discrepancies between concept definition and concept image of functions in students, keeping the students' concept image of function similar to the one from the 18th century.

Univalence. While the arbitrary nature of functions is implicit in the definition of a function, the univalence requirement, i.e., that for each element in the domain there be only one element (image) in the range, is stated explicitly. This requirement is an integral part of every curriculum that deals with functions and is emphasized in almost every text definition. For example, "Let D and R be two sets. A function from D to R is a rule that assigns to each member of D a unique member of R" (Dolciani et al., 1986).

Distinguishing between functions and non-functions by using the univalence requirement is also a very common activity in most texts on functions. For example, "State whether the given set of ordered pairs is a function. When a relation is not a function, tell why it is not" (Coxford and Payne, 1987, pp. 178–9). Similar exercises are given in most other textbooks which introduce functions (e.g., Dolciani et al., 1986; Keedy et al., 1986; Nichols et al., 1986).

While one may argue about the overemphasis this requirement gets in relation to the understanding of the concept of function and about the place in the curriculum for teaching it to students, one cannot ignore univalence completely since this is an integral part of the modern concept of function. However, knowing that functions have to be univalent is not sufficient. Secondary teachers need to know *why* functions are defined this way. They should be familiar with the historical development of functions since this explains why functions came to be the way they are defined today and provides meaning to the definition.

As the history of the development of the concept of function shows, univalence was not required at the beginning. Freudenthal (1983) attributes this requirement to the desire of mathematicians to keep things manageable. Keeping track of meanings of multivalued symbols (such as $\sqrt{\ }$), and taking care that they have the same meaning in the same context is not easy and requires a lot of watchfulness. When one has to deal with differentials of orders higher than one, one has to distinguish independent from dependent variables, and then multivalued symbols become too messy.

Advanced analysis of functions led to the restriction of functions to univalent functions only and this was generally accepted as the definition of functions.

Most of the participants in the study knew about the univalence property of functions and its use as a criterion for telling whether something was a function. Many subjects considered univalence to be important. But almost none of them could explain why it is important and how functions came to be defined that way. For example, when asked to explain the importance of univalence, one prospective teacher said: "I don't know why. I don't know why there should be one. It's the way I always learned though."

After being pressed to think of an explanation for this requirement, some subjects tried to use everyday life, engineering or science as the source of this requirement, seeing no connection to pure mathematics. Some other subjects thought that the importance of the requirement was rooted in mathematics. But the historical explanation of keeping things manageable was not always the origin for this belief. One subject, for example, described the origin of the requirement as arbitrary.

It seems like that would be, whoever decided to call that a function just made it one of the requirements. I would just think, that would be, whoever decided to call it a function just decided: if it looks like a graph, like this, and has only one, and I'm going to call that a function.

Some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence. This requirement is, usually, presented to our students as one of the most important characteristics of functions and this is what many of them think. They know that it distinguishes between relations that are not functions and those which are. But, in many cases, they are not told why is it important to distinguish between these two groups. Many mathematics teachers do not explain what is it that you can do with functions that you cannot do with relations which are not functions. This approach may contribute to making mathematics looks like an arbitrary collection of rules and definitions – an approach that the subject above seemed to hold.

Arbitrariness and univalence are the essence of the modern concept of function. Teachers, we claim, should have a good match between their concept image of a function and the *modern mathematical concept*. But this does *not* necessarily imply that they should know the *modern formal definition* of a function where a function f is defined to be any set of ordered pairs of elements such that if $(a, b)\varepsilon f$, $(c, d)\varepsilon f$, and $a = c$, then $b = d$. In other words, teachers should not necessarily think of a function as of a special subset of the Cartesian product of two sets, in which any element in the domain (the set of all first elements of the ordered pairs) should be paired with one and only one element in the range (the set of all second elements of the ordered pairs). While the pure set definition of a function is, of course, correct, it does not convey the meaning of a function as it is usually used in mathematics, science or everyday life, as was already claimed by Freudenthal (1983) and Malik (1980). In many cases, a function has the form of a correspondence, a dependence or an assignment between two sets or variables.

Different Representations of Functions

The function is described as a unifying concept. But functions appear and behave in different ways. Today functions are everywhere in mathematics. MacLane (1986) gives many examples: Algebraic operations provide examples of functions of numbers. Geometric definitions produce trigonometric functions. The exponential function and its inverse, the logarithmic function, are also numerical functions. Functions of points in the plane or in space, such as rotation, reflection and translation, arise in geometry. In group theory, the inverse is a function from the group to itself. In a metric space, the distance is a real-valued function of pairs of points. In Boolean algebra, intersection and union are functions of pairs of sets. In geometry, length is a real-valued function of curves. Computer science also provides new ways of using functions.

Freudenthal (1983) also points to the different labels functions have in mathematics: mapping, transformation, permutation, operation, functional, operator, sequence, morphism, etc. There are also different function notations which make the function concept look like different concepts instead of one unifying concept. When algebraic operations are used to describe functions, a common notation is $f: x \rightarrow 2 + x$. Many functions that were recognized as such in the early days of functions, have a special importance or are used extensively, have specific names and use specific notation. These, for example, are the trigonometric functions: sin, cos, tan, etc., and also exp and log as well as X for the random variable function in probability. Linear transformations are described, in many cases, by matrices which do not look like the common notation of functions at all. In set theory, set notation is used, such as $\langle x, y \rangle$, $\in$, $\{\ \}$, etc.

In addition to having various classes of functions, the same function can also appear in different representations. The most common representations of functions are formulae and Cartesian graphs. Other representations are arrow diagrams, tables, sets of ordered pairs, and situations from everyday life or other disciplines. In higher mathematics, functions are often represented by a symbol only. The following example illustrates the difficulties caused by a lack of connectedness between representations.

When given the following problem:

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If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

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that involved a quadratic expression in one representation (symbolic), most subjects (about 80%) tried to solve it (unsuccessfully) by using that representation only, even if the use of another representation (graphic) could have been much easier and more appropriate. The quadratic function is a very fundamental and basic function in the high school curriculum. The prospective teachers have studied about and used it since they were in high school. They were, probably, very familiar with it in both symbolic and graphic representations. Still, seeing a quadratic expression did not immediately bring to mind the graphic representation. A lack of rich relationships and connectedness between the two representations seems to prevent many of the prospective teachers from relating the given equation $ax^2 + bx + c = 0$, to a graphical representation of the function $f(x) = ax^2 + bx + c$. Eisenberg and Dreyfus (1986) report similar findings when even in a course which stressed the graphical method of solving inequalities very few of the college students opted for the graphical solution on the final exam.

Alternative Ways of Approaching Functions

Different uses of functions force us to approach functions differently. Sometimes we have to deal with functions point-wise, i.e., to plot, read or deal with discrete points of the function either because we are interested in some specific points only or since the function is defined on a discrete set. Reading values from a given graph or finding the discrete density of a discrete random variable are examples of a point-wise approach to functions. Other times we need to look at intervals, for example, when we deal with a local extremum. There are also times when we have to consider the function in a global way, and look at its behavior. For example, when we want to sketch the graph of a function given in a symbolic form or when we want to find an extremum of a function which is defined on the real numbers. And there are times when we deal with functions as entities or objects. For example, when we deal with families of functions or when we define functions of functions.

Each one of the alternative ways of approaching functions is different from the others and neither one of them is appropriate for all situations. Sometimes, when more than one way can be used, some ways are more appropriate than others. The following example illustrates the importance of choosing between the different approaches.

When asked how they would explain to a student in algebra 2 how to graph the function $f(x) = \frac{1}{x^2 - 1}$, half of the interviewees started their explanations by suggesting they make a chart of some values of x and y (usually small whole numbers and their inverses, which are "easy" to deal with), plot the points and then connect them in order to produce a smooth curve. The other half suggested looking first for undefined points – an approach that pays attention to the behavior of the function.

The first approach, which emphasizes a point-wise approach to functions, is not hard to learn. Other researchers (Bell and Janvier, 1981; Janvier, 1978; Lovell, 1971; Marnyanskii, 1975; Monk, 1988) report that many students are able to deal with functions point-wise only, i.e., they can only plot and read points, but cannot think of a function as it behaves over intervals or in a global way. But a point-wise approach to graphing functions is, in many cases, less powerful than a method that emphasizes graphing based on a more global analysis of the behavior of a function. For example, graphing a quadratic function that has $(-100, 78)$ as a vertex by plotting several points near $(0, 0)$ will not produce a very informative graph. Moreover, graphing a function that is discontinuous at $x = 0.3$ by plotting several points with whole number coordinates, and then connecting them to make a smooth curve will produce the wrong graph.

Interestingly, all the prospective teachers in the study have had calculus and other advanced courses in mathematics. So, all of them should have known that the analysis of some characteristics of the function to be graphed is important. They also should have known that some points are more important than others, so producing good graphs cannot be based on the choice of numbers that are easy to compute. Easy calculation is especially not a good argument at the end of the 1980's, when calculators and computers are so widespread. The National Council of Teachers of Mathematics (NCTM, 1989a) also recommends avoiding graphing by hand using tables of values. But it seems that several of the prospective teachers had a strong tendency to use it as the graphing method. In order for teachers to be able to help students to be flexible in their approach to functions and make good choices between the different available approaches, the teachers themselves need to have that knowledge and understanding.

Knowing what functions are and being able to work with them in different forms, representations and notations using appropriate ways, is important. But, as Freudenthal (1983) says, "the strength of the function concept is rooted in the new operations – composing and inverting functions – which create new possibilities" (p. 523). This is discussed in the next section.

The Strength of the Concept – The Inverse Function and the Composition of Functions

Functions opened new opportunities which Freudenthal (1983) considers to be the cause for the success of functions. In addition to the typically algebraic operations of addition, subtraction, multiplication, division and raising to power, functions can also be composed and inverted. The composition of functions is described as having created "a never before known wealth of new objects – functions as wild as one wants to contrive" (p. 523). The ability to substitute functions into each other and invert them created new functions and helped with the study of differentials and integrals. Freudenthal attributes that to the explosive growth of the analysis. Understanding of the concept of function must, therefore, include an understanding of the composition of functions and the inverse function. The inverse function and the composition of functions, as any other concept, cannot be understood in one simplistic way only. Understanding these sub-concepts of the concept of function requires understanding the general meaning as well as the formal mathematical definition. The following example about the limitation of understanding the inverse function as "undoing" only illustrates this point.

"Undoing" is an important meaning of the inverse function which captures the essence of the definition. The importance of this informal meaning is also recognized by the National Council of Teachers of Mathematics (NCTM, 1989a) which recommends that all students explore the concept of inverse function informally as a process of undoing the effect of applying a given function, while the precise definition of the inverse function and composition of functions be reserved for college intending students.

But understanding the inverse function as "undoing" is not enough. This term is too vague and imprecise as we can learn from the way prospective teachers answered when they were asked the following problem:

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A student said that there are 2 different inverse functions for the function $f(x) = 10^x$: One is the root function and the other is the log function. Is the student right? Explain.

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Most people used the idea of "undoing" as their interpretation of an inverse function. The xth root of 10 seemed to them to "undo" what 10^x does: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. So, the "undoing" conception of an inverse function misled many of the participants in their search of the inverse function of $f(x) = 10^x$. The "feeling" that an inverse function gives back what one started with (10 in our example, instead of x) led many subjects to wrongly conclude that the root was the inverse function of $f(x) = 10^x$. So, a solid understanding of the concept of inverse function cannot be limited to "undoing". Teachers need to have an informal conception as well as more formal knowledge.

Basic Repertoire – Functions of the High-School Curriculum

What should a basic repertoire of functions for secondary mathematics teachers include? A partial answer to this question is stated implicitly in almost every curriculum guide (e.g., Academic Preparation in Mathematics, 1985; Chambers et al., 1986; NCTM, 1989a; Michigan Essential Goals And Objectives For Mathematics Education, 1988; Oregon Mathematics Concept Paper No.2, 1987) – these are the familiar examples that students meet in high school. They include, for example, linear, quadratic and general polynomial; exponential and logarithmic; trigonometric and rational functions. They should also include some examples of functions from discrete mathematics.

Not only are these specific functions important as a basic repertoire for knowing functions, they are especially important for secondary teachers – the specific population being considered in this paper – since they are the very same functions these teachers need to teach as a basic repertoire to their students. Every high school text on mathematics which includes functions includes some or all of these specific functions. Therefore it is reasonable to assume that every high school teacher should have a good grasp of these specific functions in particular. The following example highlights the importance of having a meaningful basic repertoire.

The prospective teachers were asked the following question:

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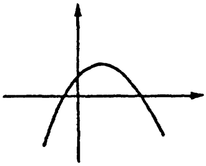
This is the graph of the function $f(x) = ax^2 + bx + c$.
State whether a, b and c are positive, negative or zero.
Explain your decision.



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Almost all of the subjects remembered that when the graph of a quadratic function looks like $\cap$, "a" (in the equation) should be negative. But many of them could not explain why the rule holds.

Okay, this is, um, basically there are some rules involving this type of equation. And, um, a lot of way it's usually taught is to memorize the rules and that's probably what I've done. And I remembered the rules and the coefficient of the x^2 term is negative and it opens downward and when it's positive it opens upwards. So this opens downward so it must be negative.

The quadratic function is a special and important case of the functions used in high school mathematics. If one understands the relationship between "a" in the quadratic expression and the graph, one has the ability to generalize to related relationships between the leading coefficient of any polynomial and its graph. Whereas, memorizing the "rule" for quadratics gives no basis for generalization. So understanding the relationship between the role of "a" in the symbolic representation of a quadratic function and in the graphic representation is very powerful. But memorization only does not empower the learner.

Related to the idea of basic repertoire, although not completely parallel, is the idea of understanding mathematical concepts and topics which are closely connected to the concept of function. One such topic is an understanding of the structure of different number sets which serve as domain and range for most of the basic repertoire functions, namely natural numbers, integers, rationals, irrationals, real numbers, and even complex numbers. Since trigonometric functions should serve as part of the basic repertoire, teachers need to have an understanding of radians. The following example illustrates an incomplete understanding of a number set structure which causes difficulties with an understanding of functions.

The subjects were asked to graph the following function:

$$g(x) = \begin{cases} x, & \text{if x is a rational number,} \\ 0, & \text{if x is an irrational number.} \end{cases}$$

Some of them seemed to approach the problem of graphing the function point-wise. They started from 0 and then tried to sketch the graph point by point, as if the set of real numbers and/or the set of irrational numbers is countable. For example, one subject sketched the graph in Fig. 2 and then explained:

1 is going to be 1, π is going to be 0. e is going to be 0, 2 is going to be 2 (pause). Well, it's going to be smooth everywhere except where you get to an irrational point and then you're going to have a sharp point which is not going to be continuous. Wherever there's an irrational number it's not going to be continuous.

It's going to curve up until it gets to, like if there's an irrational number between 0 and 1 it's going to go down to the irrational number, and then it's going to, we've got our negative numbers too. Maybe it's like this [points to the graph – see Figure 2], except it's not going to be smooth because every time you hit zero it's going to come straight down and then it will have to go to the next rational or irrational number. I'm being really general about this.

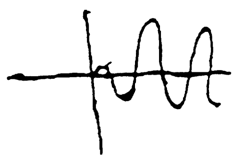

Fig. 2. "Graph" of $g(x) = \begin{cases} x, & \text{if x is a rational number,} \\ 0, & \text{if x is an irrational number.} \end{cases}$

This subject thought of the real numbers as a countable and even discrete set, as if one starts from 0 and keeps going to the *next number*; as if one goes through several numbers until one hits an irrational number. She held a misconception about the structure of real numbers which suited the point-wise approach to graphing that she used. But real numbers are used as the domain and range of many functions in the high school and college curriculum. Having a wrong conception of the structure of the sets which serve as domain and range leads to a wrong understanding of the function itself.

Knowledge and Understanding of the Function Concept

The following two examples demonstrate the importance of both procedural and conceptual knowledge and the relationships between them. The first example goes back to the problem that was already discussed in the section 'Different representations of functions', where the subjects were told that if one substitutes 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers), one gets a positive number and substituting 6 gives a negative number. The subjects were then asked to find the number of real solutions of the equation $ax^2 + bx + c = 0$. As we have already seen, lack of rich relationships and connectedness between different representations of the same function (conceptual knowledge) prevented subjects from solving the problem. Not only that, but weak conceptual knowledge also did not warn the subjects when, by misusing procedural knowledge (for solving a system of two equations: $a + b + c = 0$ and $36a + 6b + c = 0$), they got answers which do not make sense at all, such as ∞ and even 3, 4 or 5 as the number of solutions of a quadratic equation. Even though they probably "knew" that a quadratic equation has at most two solutions, the prospective teachers lacked the connections which characterize conceptual knowledge to make this knowledge accessible.

In contrast, in another example which was introduced in the section 'The strength of the concept – The inverse function and the composition of functions', the subjects were asked to decide whether a student was right in claiming that both the root function and the log function are inverse functions for the function $f(x) = 10^x$. The analysis of responses shows how a correct use of procedural knowledge can help in cases where the conceptual knowledge is naive and immature.

The most common description of the root function by the participants was the xth root of 10 or just the xth root (without specifying 'of what'). Some people then used their procedural knowledge of inverse functions and applied correctly the algorithm for checking whether a function is an inverse function by composing the two functions and checking to see if they got the identity function $f(x) = x$ as the result. For example:

$$\text{“}f^{-1}(x) = \log x : \log(10^x) = x \log 10 = x \ - \text{ correct}$$

$$f^{-1}(x) = \sqrt[x]{x} : (10^x)^{1/x} = 10 \ - \text{ incorrect.”}$$

(Note that the subject did not really check the function $f^{-1}(x) = \sqrt[x]{x}$.)

But most people did not use their procedural knowledge about inverse functions. Rather they used their naive conceptual knowledge of what an inverse function was. These people used the idea of "undoing" as their interpretation of an inverse function and wrongly concluded that root was the inverse function of $f(x) = 10^x$. Therefore, correct procedural knowledge can, sometimes, help in monitoring naive conceptual knowledge.

Knowledge about Mathematics

Knowledge about the nature of mathematics influences the substantive knowledge of functions. The use of means in a mathematically wrong way may lead to an incorrect knowledge of functions. For example, inductive and deductive reasoning are basic to mathematics. Their importance is also recognized by the National Council of Teachers of Mathematics (NCTM, 1989a) which recommends that in grades 9–12, the mathematics curriculum should include principles of inductive and deductive reasoning. Students should experience the making of a conjecture by generalizing from a pattern or observations made in particular cases (inductive reasoning) and then test the conjecture by constructing either a logical verification or a counter example (deductive reasoning).

Investigation of a situation by checking specific cases is a very powerful strategy in mathematics. Many discoveries were made by inductive reasoning. Looking at specific cases helps not only with the formulation of a generalization but also with the understanding of the situation. But inductive reasoning is not enough as an explanation for the existence of the rule. In other words, checking examples is not a proof. The following example illustrates this point. The subjects were given a graph of a quadratic function and were asked to decide what the signs of "a", "b" and "c" of the corresponding symbolic representation: $f(x) = ax^2 + bx + c$ were. Many subjects used a method of formulating rules based on the checking of a very limited number of simple quadratic functions. They "found", for example, that "b" is positive if and only if the axis of symmetry goes through a positive x.

Making conclusions which are based only on the checking of some examples without making sure that all possibilities are covered or using deductive reasoning, as was shown in the above example, points to a lack of understanding of what counts as an explanation, and which ways are considered appropriate and acceptable in mathematics for transforming a conjecture to a theorem, i.e., what is acceptable as a proof. Knowledge about the nature of mathematics is, therefore, an integral and important part of the knowledge of functions.

CONCLUSION

How do teachers construct their subject matter knowledge? A great deal of that is done throughout their K-12 study and college courses. However, as we can learn from the examples in this paper, prospective secondary teachers' knowledge of functions tends to be weak and fragile. We cannot assume that they have a comprehensive and well-articulated knowledge of the mathematics they have to teach. The same conclusion was reached by Ball (1988, 1990) in relation to elementary and secondary mathematics. Efforts must be made, therefore, to improve mathematics courses in general as well as teach prospective teachers the mathematics they have to teach. Mathematics courses should be constructed differently so that better understanding will be developed. But, in addition to their regular mathematics courses, teachers need special courses, in which they can learn mathematics for *teachers*. These courses should be developed in the light of the seven aspects of the framework described in this paper and need to deepen and integrate the subject matter knowledge these teachers need to teach.

In these courses teachers need to meet the subject content they have to teach in ways which are different from the ways they have been used to previously. Examples for that can be found in Lappan and Even (1989) and in Tirosh, Nachmias and Arcavi (1990). The first describes mathematical experiences for prospective elementary teachers in which they explore, for instance, the concept of distance in an unfamiliar geometry. The latter is intended for secondary teachers and describes a re-viewing of linear functions through an exploration of an unfamiliar graphical representation. Meeting a "familiar" concept in an unfamiliar situation forces the teachers to re-examine their subject matter knowledge, overcome difficulties, and construct a better, deeper and more articulated notion.

NOTES

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